Published by Institute of Physics Publishing for SISSA/ISAS

Received: *June 1, 2004*
Accepted: *July 19, 2004*

D-branes in topological minimal models: the Landau-Ginzburg approach

Anton Kapustin and Yi Li

California Institute of Technology, Department of Physics
Pasadena, CA 91125, U.S.A.
E-mail kapustin@theory.caltech.edu yili@theory.caltech.edu

Abstract: We study D-branes in topologically twisted $\mathcal{N} = 2$ minimal models using the Landau-Ginzburg realization. In the cases of A and D-type minimal models we provide what we believe is an exhaustive list of topological branes and compute the corresponding boundary OPE algebras as well as all disk correlators. We also construct examples of topological branes in E-type minimal models. We compare our results with the boundary state formalism, where possible, and find agreement.

Keywords: Topological Field Theories, D-branes.

Contents

1. Introduction

Topological Landau-Ginzburg models are a rich, and at the same time simple, class of topological field theories (TFTs). When these models are considered on a world-sheet without boundaries, all topological correlators are given by a simple closed formula [11]. In refs. [7, 8] we have generalized these results to world-sheets with boundaries which lie on arbitrary D-branes of type B. In this paper we work out a particular example: D-branes in topologically twisted $\mathcal{N}=2$ minimal models. $\mathcal{N}=2$ minimal models are rational $\mathcal{N}=2$ superconformal field theories. Modular-invariant combinations of characters of $\mathcal{N}=2$ super-Virasoro algebra have ADE classification, so one can talk about minimal models of type A, D, or E. These theories have Landau-Ginzburg realizations, the superpotentials being

$$\begin{aligned}
W_{A_m} &= x^{m+1} \qquad m \geq 1\,,\\
W_{D_m} &= x^{m-1} + xy^2 \qquad m \geq 4\,,\\
W_{E_6} &= x^3 + y^4\,,\\
W_{E_7} &= x^3 + xy^3\,,\\
W_{E_8} &= x^3 + y^5\,.
\end{aligned}$$

$\mathcal{N}=2$ minimal models admit a B-twist, so one can study associated TFTs and their D-branes. This is what we will do in this paper. Prior work in this direction includes refs. [2] and [4]–[6].[1]

One can also study D-branes in the untwisted minimal models using methods of boundary conformal field theory. The simplest D-branes are those which preserve the full chiral algebra; these are so-called Cardy branes. In fact, since $\mathcal{N}=2$ super-Virasoro algebra admits a non-trivial automorphism (the mirror involution), there are two kinds of Cardy branes, which are exchanged by the mirror involution. They are known as A-branes and B-branes, because they are similar to A and B-branes on Calabi-Yau manifolds. Cardy B-branes are precisely D-branes compatible with the topological B-twist, so one can study D-branes in topologically twisted minimal models using the formalism of boundary conformal field theory (the boundary state formalism). For minimal models of type A, such analysis has been performed in ref. [9], and we will compare our results with those of ref. [9] below. It should be stressed that our methods enable one to compute all possible topological correlators, including those on Riemann surfaces with arbitrary number of handles and holes. This is very hard to do in the boundary state formalism, where so far only certain disk correlators have been computed. On the other hand, the boundary state formalism allows one to access non-topological correlators, which are beyond the reach of our methods.

To set the stage, let us summarize the main results of ref. [11] and refs. [7, 8]. Related results have also appeared in refs. [6, 4]. First, let us recall how to compute bulk topological correlators in a LG model with superpotential W. To describe them, it is sufficient to specify the ring of bulk observables and the one-point function on a genus-g Riemann surface for all g. The bulk ring is given by

$$\frac{\mathcal{O}}{\partial W},$$

where $\mathcal{O}$ is the ring of holomorphic functions on the target space X. This is known as the Jacobi ring of W. In what follows we will assume that $X=\mathbb{C}^n$. We also assume that W is polynomial and replace the ring of holomorphic functions with the ring of polynomial functions $\mathbb{C}[z_1,\ldots,z_n]$. Finally, we will assume that all critical points of W are isolated. In this case the Jacobi ring is finite-dimensional.

Let α be a polynomial representing an element of the Jacobi ring. The corresponding one-point function on a genus-g Riemann surface is given by

$$\langle\alpha\rangle_g=\frac{1}{(2\pi i)^n}\oint\frac{\alpha H^g}{\partial_1 W\,\partial_2 W\ldots\partial_n W},$$

where H is the hessian of W. In this formula the integrand is regarded as a meromorphic n-form (i.e. a factor $dz_1\wedge\ldots\wedge dz_n$ is implied), and the integral is performed over an n-dimensional real submanifold defined by the equations $|\partial_i W|=\epsilon_i$, where all ϵ_i are small [11]. For sufficiently small ϵ_i this submanifold is a union of several lagrangian tori each of which

[1]While this paper was in preparation, there appeared ref. [1], which also discusses D-branes in A-type minimal models using the approach of ref. [7].

JHEP07(2004)045

encloses a single critical point of W. Alternatively, we may rewrite this expression as an integral of a certain form of type $(n, n-1)$ over a large $2n-1$-dimensional sphere in $\mathbb{C}^n$ [3]. The possibility of such a rewriting expresses the fact that the integral is a kind of multi-dimensional residue, and can be evaluated using only the behavior of the integrand at infinity. The above formula is derived by evaluating the path-integral in the zero-mode approximation, which can be argued to be adequate in the topological sector.

Second, let us describe topological D-branes and the corresponding correlators. Topological D-branes are localized at the critical points of W, and D-branes sitting at different critical points do not "talk" to each other. Thus we may focus on any one critical point. Without loss of generality, we may assume that the critical point is at $z=0$, and $W(0)=0$. Then D-branes sitting at $z=0$ correspond to ways of factorizing of W into a product of two matrix polynomials:

$$d_0 d_1 = d_1 d_0 = W \cdot id\,, \qquad d_0, d_1 \in \mathrm{Mat}(r, \mathbb{C}[z_1, \ldots, z_n]).$$

This was proposed by M. Kontsevich (unpublished). A physical derivation of Kontsevich's proposal has been given in refs. [7, 8].

Given such matrix factorization, we can describe boundary operators as follows. Consider a $2r \times 2r$ matrix polynomial

$$Q = \begin{pmatrix} 0 & d_1 \\ d_0 & 0 \end{pmatrix}.$$

It can be regarded as a linear operator on the vector space M of $2r$-dimensional vectors with polynomial components. M has a natural $\mathbb{Z}_2$ grading: the first r components are declared even, while the last r are declared odd. The operator Q is odd, in the sense that it maps the even part of M to its odd part, and vice versa. Q satisfies $Q^2 = W \cdot id$. In mathematical terminology, M is a free module of rank $2r$ over the algebra of polynomial functions $\mathcal{O}$. In ref. [8], the pair $(\mathcal{O}, W)$ is called a CDG algebra, while the pair (M, Q) as above is called a free CDG module over the CDG algebra. The origin of this terminology is explained in ref. [8].

Now consider the space P of all polynomial linear operators on M which commute with multiplication by polynomial functions. This is simply the space of $2r \times 2r$ matrices with polynomial entries, and it has an obvious grading. Define a linear operator on P using the "adjoint" action of Q:

$$\mathcal{D} : a \mapsto [Q, a] = Qa - (-1)^{|a|} aQ\,, \qquad a \in P\,.$$

Here $|a| = 0$ or 1 depending on whether a is even or odd. It is easy to see that $\mathcal{D}$ satisfies $\mathcal{D}^2 = 0$. It was shown in ref. [8] that the cohomology of $\mathcal{D}$ coincides with the space of physical boundary operators. This space is graded, and it also has the structure of a ring (because one can multiply matrices). This is nothing but the ring of boundary operators. Thus the boundary ring is completely determined by the matrix Q. One can show that although P is infinite-dimensional, the cohomology of $\mathcal{D}$ is finite-dimensional, as expected

JHEP07(2004)045

on physical grounds. In mathematical terms, $\mathcal{D}$ cohomology computes the endomorphism algebra of a CDG module (in the derived category of CDG modules). Therefore we will use the expressions "boundary OPE algebra" and "endomorphism algebra" interchangeably.

Given a pair of CDG modules, one can also compute the space of boundary-changing operators for the corresponding pair of branes [8]. This space can be identified with the space of morphisms in the derived category of CDG modules. Multiplication of boundary-changing operators corresponds to the composition of morphisms.

Finally, one can write down a closed formula for arbitrary bulk-boundary topological correlators. Consider a Riemann surface with g handles and h holes. Suppose the boundary of the i^{th} hole is mapped to a D-brane associated with a matrix polynomial Q_i. Clearly, it is sufficient to consider the case when there is one bulk operator insertion α and h boundary operator insertions ϕ_i, one on each boundary circle. Here α is an element of the Jacobi ring, and ϕ_i is a class in the cohomology of $\mathcal{D}_i$. We have shown [8] that the corresponding correlator is given by

$$\frac{1}{(n!)^h(2\pi i)^n}\oint \frac{\alpha H^g}{\partial_1 W\partial_2 W\ldots\partial_n W}\cdot\prod_{i=1}^{h}\mathrm{STr}\left[(\partial Q_i)^{\wedge n}\phi_i\right].$$

Here we identify top forms like $(\partial Q_i)^{\wedge n}$ with functions, and an overall factor $dz_1\wedge\ldots\wedge dz_n$ is implied, so the integrand is a meromorphic n-form. The integration is over a union of lagrangian tori enclosing all the critical points of W, as before. This integral can be regarded as a generalized residue, and can also be rewritten as an integral of a $2n-1$-form over a large $2n-1$-dimensional sphere in $\mathbb{C}^n$ [3]. Some special cases of this formula have been derived in refs. [6, 4].

2. D-branes in $\mathcal{N}=2$ minimal models

In this section we classify topological B-branes in $\mathcal{N}=2$ minimal models using the methods described above. Previous works which use the Landau-Ginzburg viewpoint to study B-branes in minimal models include refs. [1] and [4]-[6]. In these papers only A-type minimal models have been discussed. Our computation agrees with the previously obtained results and clarifies some additional subtleties. We also provide what we think is a complete list of irreducible B-branes in the case of D-type superpotential. We conjecture that any other B-brane for the D-type superpotential is a direct sum of the ones we have constructed. Finally we give examples of B-branes in E-type minimal models.

2.1 A-type minimal models

As mentioned above, $\mathcal{N}=2$ minimal models of type A are believed to be the infrared fixed points of Landau-Ginzburg models with superpotential $W_{A_m}=z^{m+1}$. Before going into detailed calculations, we pause to comment on an important subtlety. Naively, adding a massive chiral field has no effect in the infrared. This is obviously true in the closed string sector (the Jacobi ring is unaffected by the addition of squares). In the open string sector, however, there is a nontrivial effect. As we shall see, adding a single massive chiral field leads to a different D-brane spectrum. What is the interpretation of this in CFT terms?

JHEP07(2004)045

Any $\mathcal{N}=1$ SCFT has a $\mathbb{Z}_2$ symmetry which acts non-trivially only on the left-moving Ramond-sector states. In the context of string theory, this symmetry is usually called $(-1)^{F_L}$. Orbifolding by this $\mathbb{Z}_2$ gives a new $\mathcal{N}=1$ SCFT which has a different spectrum of D-branes. If the original theory has $\mathcal{N}=2$ superconformal symmetry, so will the orbifolded one. We claim that adding an extra massive chiral field to the LG superpotential has exactly the same effect on D-branes as orbifolding by this $\mathbb{Z}_2$ symmetry. To test this claim, note that the orbifolded model also has a $\mathbb{Z}_2$ symmetry, which acts by reversing the sign of all twisted sector states, which are all in the Ramond sector. Orbifolding by this $\mathbb{Z}_2$ gives the original (unorbifolded) CFT. On the LG side, the second orbifolding corresponds to adding yet another free massive chiral superfield, therefore we expect that adding two squares to W has no effect on D-branes. This is indeed true, for arbitrary W [10]. Below, we will verify our claim in the case of A-type minimal models.

With this ambiguity in mind, we will classify in the following the B-branes for both $W=z^n$ and $W=z^n+y^2$. We will see in the next section that they match up precisely with the Cardy branes in the minimal model and its $\mathbb{Z}_2$ orbifold. The D-brane spectrum for $W=z^n$ has already appeared in the literature [10, 1], so we start by summarizing this case first. As explained in ref. [7, 8] and reviewed in the Introduction, B-branes are classified by CDG modules (E,Q) over the $\mathbb{Z}_2$-graded CDG algebra $(\mathcal{O},W)$. In our case $\mathcal{O}$ is simply the algebra of polynomials in a single variable z. We have $k+1$ obvious solutions to the equation $Q^2=W$:

$$E_k=\left\{\mathcal{O}\oplus\mathcal{O},\, Q=\begin{pmatrix}0 & z^k\\ z^{n-k} & 0\end{pmatrix}\right\}, \tag{2.1}$$

where $k=0,\ldots,n$. Recall that the BRST operator acts on $P_k\simeq \mathrm{Mat}(2,\mathbb{C}[x])$ as

$$\mathcal{D}:\ \phi\mapsto[Q,\phi]$$

and the space of endomorphisms (topological open strings) on the brane E_k is given by the $\mathcal{D}$-cohomology. The cases $k=0$ and $k=n$ actually turn out trivial: the $\mathcal{D}$ cohomology vanishes, as one can easily check.[2] In mathematical terms, this means that E_0 and E_n are zero objects in the category of B-branes. Interesting branes correspond to the range $0<k<n$. In fact one can further restrict the range to $0<k\le n/2$ for the following reason. To any brane one can associate its anti-brane by exchanging d_0 and d_1 and flipping the grading on E. We will call this operation parity-reversal and will denote the parity-reversal of E by $\bar{E}$. Clearly, the space of topological strings between a brane E and some other brane E' is the same as the space of topological strings between $\bar{E}$ and E', except for the reversal of grading. In our case parity-reversal amounts to $k\mapsto n-k$. Thus it is sufficient to deal with k in the range $0<k\le n/2$. A case of special importance is when n is even and $k=n/2$. This brane is its own anti-brane.

[2] Alternatively, this can be seen from the path-integral derivation of Kontsevich's proposal in ref. [8], since the path integral localizes at $Q=0$. For $k=0$ or $k=n$ this equation has no solutions, and the path integral identically vanishes.

JHEP07(2004)045

It is straightforward to check that for $k \leq n/2$ the space of endomorphisms of E_k is spanned by k even elements

$$a_i = \begin{pmatrix} z^i & 0 \\ 0 & z^i \end{pmatrix}, \qquad i = 0, 1, \ldots, k-1$$

and k odd elements

$$\eta_i = \begin{pmatrix} 0 & z^i \\ -z^{n-2k+i} & 0 \end{pmatrix}, \qquad i = 0, 1, \ldots, k-1 .$$

The OPE algebra is simply given by matrix multiplication, modulo the image of $\mathcal{D}$. We can describe this algebra more compactly by giving its generators and relations. There is one even generator $a = a_1$ and one odd generator $\eta = \eta_0$, and the relations are

$$\eta a = a \eta\,, \qquad a^k = 0\,, \qquad \eta^2 = -a^{n-2k}\,. \tag{2.2}$$

Note that for $k \leq n/3$ the second relation is equivalent to $\eta^2 = 0$.

Morphisms between two different branes, E_i and E_j, can also be calculated without difficulty, as explained in ref. [8]. Here we merely quote the results. As explained above, one may assume $i, j \leq n/2$. For $i \leq j$, the space of morphisms from E_i to E_j is spanned by i even elements

$$a_k = \begin{pmatrix} z^{j-i+k} & 0 \\ 0 & z^k \end{pmatrix}, \qquad k = 0, 1, \ldots, i-1$$

and i odd elements

$$\eta_k = \begin{pmatrix} 0 & z^k \\ -z^{n-i-j+k} & 0 \end{pmatrix}, \qquad k = 0, 1, \ldots, i-1.$$

The results for the case $i > j$ are obtained from these by parity reversal. Composition of morphisms is given by matrix multiplication, modulo the image of $\mathcal{D}$.

In addition to the branes E_k, $k \leq n/2$, we also have branes E_k, $k \geq n/2$. Since E_{n-k} is the anti-brane of E_k, no separate analysis of these branes is required. Note that the brane E_k is not isomorphic to its anti-brane E_{n-k}. Indeed, morphisms from E_k to E_{n-k} are the same as morphisms from E_k to E_k, except for parity reversal. If E_k were isomorphic to E_{n-k}, then there would be an invertible even morphism between them, or equivalently, there would be an invertible odd morphism from E_k to itself. One can easily see that there are no such morphisms (all odd endomorphisms are nilpotent). The case $k = n/2$ is an exception, because the odd endomorphism η is invertible, its inverse being $-\eta$. Thus the brane $E_{n/2}$ is isomorphic to its own anti-brane. We will express these facts by saying that the branes E_k, $k < n/2$ are orientable, while $E_{n/2}$ is unorientable.

It is shown in ref. [10] (see also comments in ref. [8]) that any topological D-brane is isomorphic to a sum of the branes E_k for some k. Thus we have a complete classification of B-branes in the A-type minimal models.

Before moving on, we take a closer look at a special example: the "fundamental" brane E_1. In this simplest case the algebra of endomorphisms is two-dimensional and spanned by

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \eta = \begin{pmatrix} 0 & 1 \\ -z^{n-2} & 0 \end{pmatrix}.$$

The only non-trivial algebraic relation of the OPE is

$$\eta \cdot \eta = \begin{cases} -1, & n = 2 \\ 0, & n > 2 \end{cases}$$

For $n > 2$, the boundary OPE algebra is simply the exterior algebra $\wedge^*(V)$, where V is a one-dimensional vector space. On the other hand, the OPE algebra for $n = 2$ is the Clifford algebra $\mathrm{Cl}(1, \mathbb{C})$. The case $n = 2$ is special because the LG model is massive, and the appearance of Clifford algebras in the boundary OPE is a generic feature of massive LG theories [7].

Next we move on to the case $W = x^n - y^2$. There are two main differences compared to the case studied above. First, $\mathcal{O}$ is now the algebra of polynomials in two variables x and y. Second, irreducible CDG modules in general have rank four rather than two, so the relevant space of endomorphisms is $\mathrm{Mat}(4, \mathcal{O})$. Barring a subtlety that arises when $n \in 2\mathbb{Z}$, which we shall discuss in detail below, irreducible B-branes in this theory are represented by

$$E_{k,\alpha} = \left\{ \mathcal{O}_+^2 \oplus \mathcal{O}_-^2 , \;\; Q = \begin{pmatrix} 0 & d_1 \\ d_0 & 0 \end{pmatrix} \right\} \tag{2.3}$$

where the "$\pm$" signs denote the $\mathbb{Z}_2$ grading, and

$$d_0 = \begin{pmatrix} x^k & \alpha \\ -\beta & -x^{n-k} \end{pmatrix}, \qquad d_1 = \begin{pmatrix} x^{n-k} & \alpha \\ -\beta & -x^k \end{pmatrix}, \qquad \alpha\beta = y^2 .$$

A priori, k runs from 0 to n, and $\alpha \in \{1, y, y^2\}$.[3] As before, the localization principle tells us that the cases $k = 0, n$ or $\alpha = 1, y^2$ give trivial branes. Alternatively, one can check the triviality of these objects by explicitly computing their spaces of open string states using the algorithm described below. From this point on, we shall assume $\alpha = y$ and $k \neq 0, n$. Furthermore, it is also clear that E_k and E_{n-k} define the same object in the category of B-branes, since they are related to each other by an automorphism which preserves the $\mathbb{Z}_2$ grading. The independent parameter range for irreducible branes can thus be limited to $k = 1, \ldots, [n/2]$. In fact, if $n \in 2\mathbb{Z}$, the object $E_k, k = n/2$ is reducible, as discussed below, so we restrict ourselves to

$$k = 1, 2, \ldots, \left[\frac{n-1}{2}\right], \qquad \alpha = y .$$

[3]Although there is a sign ambiguity for α, one can change the sign by conjugating d_0 and d_1 with an appropriate matrix, therefore changing the sign from plus to minus gives an isomorphic CDG module.

JHEP07(2004)045

To compute the OPE, one notes that the boundary BRST operator acts on the space of endomorphisms, which is isomorphic to $\mathrm{Mat}(4, \mathbb{C}[x, y])$, as follows

$$\mathcal{D}: \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} Bd_0 + d_1 C & -Ad_1 + d_1 D \\ -Dd_0 + d_0 A & Cd_1 + d_0 B \end{pmatrix},$$

where A, B, C, D are 2×2 matrices with values in $\mathbb{C}[x, y]$. The $\mathcal{D}$-closedness amounts to two independent conditions

$$WA = d_1 D d_0, \qquad WC = -d_0 B d_0$$

which enable one to solve for A and C in terms of B and D. After modding out $\mathcal{D}$-exact elements, we can describe $\mathcal{D}$-cohomology using B and D only. The results are as follows. Even elements (bosonic open string states) are given by

$$D \in \mathrm{End}(\mathcal{O}_-^2) / \left(\mathrm{Im}\, d_0^L \oplus \mathrm{Im}\, d_1^R\right)$$

which satisfy the divisibility condition

$$W \,\big|\, d_1 D d_0 \,.$$

Here the superscripts on d_0 and d_1 indicates whether the operator acts from left or right. Similarly, odd elements (fermionic open string states) are given by

$$B \in \mathrm{Hom}\left(\mathcal{O}_-^2, \mathcal{O}_+^2\right) / \left(\mathrm{Im}\, d_1^R \oplus \mathrm{Im}\, d_1^L\right)$$

which satisfy

$$W \,\big|\, d_0 B d_0 \,.$$

Let us first look at the bosonic sector. For the brane E_k, one can show that even elements of the endomorphism algebra can be represented in the quotient by

$$D \in \begin{pmatrix} \mathbb{C}[x]/x^k & \mathbb{C}[x, y]/(x^n - y^2) \\ \mathbb{C}[x]/x^{n-k} & \mathbb{C}[x]/x^{n-k} \end{pmatrix}.$$

The divisibility condition imposes a further relation

$$x^n - y^2 \,\big|\, y(D_{22} - D_{11}) - x^k D_{21} + x^{n-k} D_{12} \,.$$

This gives $2k$-dimensional even subspace, spanned by the following elements in $\mathrm{End}(\mathcal{O}_-^2)$:

$$a_i = \begin{pmatrix} x^i & 0 \\ 0 & x^i \end{pmatrix}, \qquad a_{k+i} = \begin{pmatrix} 0 & x^i \\ x^{n-2k+i} & 0 \end{pmatrix}, \qquad i = 0, 1, \ldots, k-1 \,.$$

Similarly, one can show that odd elements live in the quotient

$$B \in \begin{pmatrix} \mathbb{C}[x, y]/(x^n - y^2) & \mathbb{C}[x, y]/x^k \\ \mathbb{C}[x]/x^k & \mathbb{C}[x]/x^k \end{pmatrix} \subset \mathrm{Hom}\left(\mathcal{O}_-^2, \mathcal{O}_+^2\right)$$

and satisfy the relation

$$x^n - y^2 \,\big|\, x^{n-k} B_{22} - x^k B_{11} + y(B_{12} - B_{21})\,.$$

Therefore the odd subspace of the boundary OPE algebra is spanned by the following elements in $\mathrm{Hom}(\mathcal{O}_-^2, \mathcal{O}_+^2)$:

$$\eta_i = \begin{pmatrix} 0 & x^i \\ x^i & 0 \end{pmatrix}, \qquad \eta_{k+i} = \begin{pmatrix} x^{n-2k+i} & 0 \\ 0 & x^i \end{pmatrix}, \qquad i = 0, 1, \ldots, k-1\,.$$

The full boundary OPE algebra can be described by generators and relations as follows. It has one even generator $a = a_1$, which is central, and two odd generators $\xi = \eta_0$ and $\eta = \eta_k$, which satisfy the relations

$$a^k = 0\,, \qquad \xi^2 = 1\,, \qquad \eta^2 = -a^{n-2k}\,, \qquad \xi\eta + \eta\xi = 0\,. \tag{2.4}$$

Obviously this algebra does not decompose as a direct sum, so the branes E_k are all irreducible. We also see that for all k there is an invertible odd endomorphism ξ. As discussed above, this means that the branes E_k are unorientable (are isomorphic to their own anti-branes).

The case $n \in 2\mathbb{Z}$ and $k = n/2$ is somewhat special in that the object $E_{n/2}$ is reducible. In fact, it splits into two irreducible branes corresponding to the following CDG modules:

$$E_\pm = \left\{ \mathcal{O}_+ \oplus \mathcal{O}_-\,,\ Q_\pm = \begin{pmatrix} 0 & x^{n/2} \pm y \\ x^{n/2} \mp y & 0 \end{pmatrix} \right\}.$$

Clearly E_- is the anti-brane of E_+.

It is a simple matter to determine the boundary OPE algebras. The endomorphism algebra of $E_\pm$ is spanned by $n/2$ even elements:

$$a_i = \begin{pmatrix} x^i & 0 \\ 0 & x^i \end{pmatrix}, \qquad i = 0, 1, \ldots, \frac{n}{2} - 1.$$

In contrast to branes E_k discussed above, there are no odd elements. This algebra has one generator $a = a_1$ and a relation $a^{n/2} = 0$. One can likewise work out the space of boundary changing operators living in $\mathrm{Hom}(E_+, E_-)$. It is purely odd and has dimension $n/2$. This implies that the branes E_+ and E_- are not isomorphic, and therefore both are orientable.

Note that in all previous examples the Witten index of the space of topological strings from a brane to itself was zero. For the branes $E_\pm$ the Witten index is equal to $n/2$.

2.2 Orbifolded A-type minimal models

For any Landau-Ginzburg model the category of D-branes has an obvious symmetry (autoequivalence): parity-reversal, which takes branes to their anti-branes. We may consider orbifolding the category by this symmetry. Note that this symmetry does not act on the target space of the LG model, but only on the category of D-branes. Such symmetries are often called *quantum* symmetries. From the string theory point of view, these are symmetries of the world-sheet theory which do not originate from any symmetry of the target space. A well-known example of a quantum symmetry is T-duality, which, in its most basic form, states that the quantum sigma-model whose target is a torus with a flat metric is unchanged if one replaces the torus with its dual. The mathematical counterpart of this phenomenon is the Fourier-Mukai transform which identifies the derived categories of an abelian variety and its dual. The situation in our case is similar, but simpler. The physical counterpart of parity-reversal is a non-geometric $\mathbb{Z}_2$ symmetry of the SCFT which acts trivially on the NS-sector states and by -1 on all RR-sector states (there are no mixed NS-R states in our case). Since chiral primary states reside in the NS sector, the chiral rings of the original and orbifolded theories are identical. In other words, orbifolding has no effect on topological closed-string states. But the properties of Cardy branes in the two theories are different, as was shown in ref. [9]. Therefore we expect that the orbifolded category of topological branes is different from the unorbifolded one. We will see that this is indeed the case for $W = z^n$. Moreover, we will see that topological D-branes in the orbifolded category are the same as topological D-branes in the LG model $W = z^n + y^2$. This lends credence to our conjecture that adding a square to W has the same effect on the category of B-branes as orbifolding by parity-reversal.

The construction of the orbifolded category was explained in the end of ref. [8]. Rephrasing this construction a little, we get the following. Objects are pairs (E, β), where E is a D-brane in the original (unorbifolded) theory, and β is an odd endomorphism of E satisfying $\beta^2 = 1$. Morphisms (boundary-changing operators) between (E, β) and (E', β') are morphisms from E to E' intertwining β and β':

$$\phi\beta = (-1)^{|\phi|}\beta'\phi, \quad \phi \in \mathrm{Hom}(E, E')\,.$$

Note that given an object (E, β) we have another valid object $(E, -\beta)$. This other object may or may not be isomorphic to (E, β). We will see examples of both possibilities below.

Let us apply this construction to D-branes in the A-type minimal model $W = z^n$. Clearly, for β to exist, E must be isomorphic to its own anti-brane. Irreducible branes E_k in the model $W = z^n$ are not their own anti-branes, except in the case $k = n/2$. Thus we are forced to consider direct sums $F_k = E_k \oplus \bar{E}_k = E_k \oplus E_{n-k}$. Clearly, it is sufficient to take k in the range $0 < k < n/2$ (the case $k = n/2$ is special and will be considered separately). The endomorphism algebra of F_k can be inferred from the known answer for the endomorphism algebra of E_k, and we see that the only invertible odd endomorphism is, up to a scalar multiple,

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where we write $\beta \in \mathrm{End}(E_k \oplus \bar{E}_k)$ as a block matrix with elements in $\mathrm{End}(E_k)$, $\mathrm{Hom}(\bar{E}_k, E_k)$, $\mathrm{Hom}(E_k, \bar{E}_k)$, and $\mathrm{End}(\bar{E}_k)$. The scalar multiple is fixed to be ± 1 by the requirement $\beta^2 = 1$. Finally, it is easy to see that the branes corresponding to the two possible signs are actually isomorphic, an isomorphism being

$$\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the same notation. The endomorphism algebra of F_k in the orbifolded category consists of endomorphisms of $E_k \oplus \bar{E}_k$ in the unorbifolded category which supercommute with β, i.e. have the form

$$\phi = \begin{pmatrix} A & B \\ (-1)^{|B|}B & (-1)^{|A|}A \end{pmatrix}.$$

Here A is an arbitrary element of $\mathrm{End}(E_k)$, B is an arbitrary element of $\mathrm{Hom}(\bar{E}_k, E_k)$, and $|A|$ and $|B|$ denote the parities of A and B.

It is easy to check that the number and endomorphism algebras of the branes F_k agree with the number and endomorphism algebras of the rank-four branes in the LG model $W = x^n - y^2$. This confirms our conjecture that orbifolding by parity-reversal is equivalent to adding a square to the superpotential.

It remains to consider the case $k = n/2$. The brane $E_{n/2}$ in the model $W = z^n$ is isomorphic to its own anti-brane, so all we have to do to complete it to an object of the orbifolded category is to find a suitable β. From the results of the previous subsection we know that up to a scalar multiple there is only one invertible odd endomorphism of $E_{n/2}$, which we denoted η. The requirement $\beta^2 = 1$ tells us that $\beta = \pm i\eta$, so we get two possible objects $F_+ = (E_{n/2}, i\eta)$ and $F_- = (E_{n/2}, -i\eta)$. Unlike in the previous case, these two objects are *not* isomorphic. Indeed, the space of morphisms from F_+ to F_- consists of endomorphisms of $E_{n/2}$ which anti-super-commute with η. All such endomorphisms are odd, and so F_+ cannot be isomorphic to F_-. Instead, there is an odd invertible morphism from F_+ to F_- given by η itself, and this means that F_- is the anti-brane of F_+. The endomorphism algebra of F_+ consists of endomorphisms of $E_{n/2}$ which super-commute with η. It is easy to see that the space of such endomorphism is purely even and spanned by a^p, $p = 0, \ldots, n/2 - 1$ in the notation of the previous subsection. It is also easy to see that the properties of $F_\pm$ match the properties of the branes $E_\pm$ in the model $W = x^n - y^2$.

2.3 D-type minimal models

The D-type minimal models have the following LG realization:

$$W_{D_{n+2}} = x^{n+1} - xy^2\,.$$

As always, one may add another free massive chiral field z to the theory, modifying the superpotential by z^2. We will comment on the effect of this at the end of this section. For the moment, we focus on the above superpotential. The most obvious way to factorize W gives the following brane:

$$E = \left\{ \mathcal{O}_+ \oplus \mathcal{O}_-,\ Q = \begin{pmatrix} 0 & x \\ x^n - y^2 & 0 \end{pmatrix} \right\}.$$

Swapping x and $x^n - y^2$ gives the anti-brane of E. The endomorphism algebra of E is spanned by two even elements

$$a_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a_1 = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}.$$

The element a_0 is the identity, while a_1 satisfies $a_1^2 = 0$ in the $\mathcal{D}$-cohomology. Thus the boundary OPE algebra is isomorphic to the exterior algebra $\wedge^*(\mathbb{C})$ as an ungraded algebra. The anti-brane $\bar{E}$ has the same endomorphism algebra, while the space of morphism from E to $\bar{E}$ is two-dimensional and purely odd. This implies that E is not isomorphic to $\bar{E}$, and therefore the brane E is orientable.

For even n, there are two additional branes given by

$$\begin{aligned} E_1 &= \left\{ \mathcal{O}_+ \oplus \mathcal{O}_-,\ Q = \begin{pmatrix} 0 & x(x^{n/2} - y) \\ x^{n/2} + y & 0 \end{pmatrix} \right\}, \\ E_2 &= \left\{ \mathcal{O}_+ \oplus \mathcal{O}_-,\ Q = \begin{pmatrix} 0 & x(x^{n/2} + y) \\ x^{n/2} - y & 0 \end{pmatrix} \right\}. \end{aligned}$$

The space of endomorphisms of E_1 is purely even and spanned by the following $n/2+1$ elements:

$$a_i = \begin{pmatrix} x^i & 0 \\ 0 & x^i \end{pmatrix}, \qquad i = 0, 1, \ldots, \frac{n}{2}.$$

The boundary OPE algebra is is generated by $a = a_1$ with the relation $a^{n/2} = 0$. The brane E_2 has the same OPE algebra. It is easy to see that $\bar{E}_i$ is not isomorphic to E_i, and therefore both E_1 and E_2 are orientable.

Since the OPE algebras of E_1 and E_2 are identical, one naturally asks whether they define the same B-brane. To answer this question one needs to work out the space of boundary changing operators. By using the now hopefully familiar algorithm, one can show that the space of boundary changing operators living in $\mathrm{Hom}(E_1, E_2)$ is purely odd and is spanned by

$$\eta_i = \begin{pmatrix} 0 & -x^{i+1} \\ x^i & 0 \end{pmatrix}, \qquad i = 0, 1, \ldots, \frac{n}{2} - 1.$$

Since the dimension of this space is one less than the dimension of the space of endomorphisms on E_1, one concludes that E_2 is isomorphic neither to E_1 nor to the anti-brane of E_1. Therefore E_1 and E_2 define two different irreducible B-branes.

There are other branes which are not included in the above analysis. In order to see these extra branes, one needs to consider CDG modules of rank four. Specifically, we consider the following CDG modules:

$$E_{k,\alpha} = \left\{ \mathcal{O}_+^2 \oplus \mathcal{O}_-^2,\ \ Q = \begin{pmatrix} 0 & d_1 \\ d_0 & 0 \end{pmatrix} \right\},$$

where

$$d_0 = \begin{pmatrix} x^k & \alpha \\ -\beta & -x^{n+1-k} \end{pmatrix}, \qquad d_1 = \begin{pmatrix} x^{n+1-k} & \alpha \\ -\beta & -x^k \end{pmatrix}, \qquad \alpha\beta = xy^2.$$

Apart from a few slight differences, the situation here is very much like in the case $W = x^n - y^2$ analyzed above. As before, the cases of $k = 0, n+1$ or $\alpha = \pm 1, \pm xy^2$ give zero objects, so we exclude them from our list. The first difference compared to the case $W = x^n - y^2$ comes from the fact that replacing either $k \leftrightarrow n+1-k$ or $\alpha \leftrightarrow \beta$ does not give the same brane, but the anti-brane. In other words, all these branes are orientable while in the case $W = x^n - y^2$ they are unorientable. The second difference is that there appear to be more types of branes. Naively, there seems to be many different ways to choose α and β so that $\alpha\beta = xy^2$. However, changing the signs of both α and β can be undone by an automorphism which does not change the grading. It is also easy to see that exchanging α and β is equivalent to replacing a brane with its anti-brane. Thus we can restrict to the following range:

$$k = 1, 2, \ldots, \left[\frac{n+1}{2}\right] \qquad \beta = x, y\,.$$

These branes, together with their anti-branes, exhaust all the objects $E_{k,\alpha}$ defined above.

The rest of the calculation is essentially the same as before, and we summarize the results below.

The case $\beta = x$. The even generators of the space of morphisms live in the quotient space $\mathrm{End}(\mathcal{O}_-^2)/\left(\mathrm{Im}\, d_0^L \oplus \mathrm{Im}\, d_1^R\right)$. They can be represented by the following matrices:

$$\begin{pmatrix} a & \mu \\ \nu & b \end{pmatrix}, \qquad a, b \in \mathbb{C}[y]/(y^2),\ \nu \in \mathbb{C}[y],\ \mu \in \mathbb{C}[x,y]/(x^n - y^2)\,. \tag{2.5}$$

The divisibility condition leads to two independent relations:

$$x^n - y^2 \,\big|\, x^{k-1}(a-b) - \mu + x^{2k-2}\nu, \qquad x \,|\, \nu\,.$$

It follows from these relations that the even part of the endomorphism algebra is spanned by the following elements in $\mathrm{End}(\mathcal{O}_-^2)$:

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a_2 = y \cdot a_1\,, \qquad a_3 = \begin{pmatrix} 1 & x^{k-1} \\ 0 & 0 \end{pmatrix}, \qquad a_4 = y \cdot a_3\,.$$

The odd part of $\mathcal{D}$-cohomology can also be parametrized by matrices of the form of eq. (2.5). The divisibility condition now implies

$$x^n - y^2 \,\big|\, x^{k-1}a - x^{n-k}b - \mu\,, \qquad \nu = 0\,.$$

Thus the odd subspace is four-dimensional and is spanned by

$$\eta_1 = \begin{pmatrix} 1 & x^{k-1} \\ 0 & 0 \end{pmatrix}, \qquad \eta_2 = y \cdot \eta_1\,, \qquad \eta_3 = \begin{pmatrix} 0 & -x^{n-k} \\ 0 & 1 \end{pmatrix}, \qquad \eta_4 = y \cdot \eta_3\,.$$

The resulting algebra of endomorphisms can be described by generators and relations. There are two even generators $a = a_2$ and $b = a_3$, two odd generators η_1 and η_3, and the following relations:

$$\begin{aligned} a^2 &= 0\,, \qquad b^2 = b\,, \qquad \eta_1^2 = 0\,, \qquad \eta_3^2 = 0\,, \qquad \eta_1\eta_3 + \eta_3\eta_1 = -1\,, \\ b\eta_1 &= 0\,, \qquad b\eta_3 = \eta_3\,. \end{aligned}$$

JHEP07(2004)045

In addition, a and b commute with everything. It is not hard to see that this algebra is isomorphic to the endomorphism algebra of the object $E \oplus \bar{E}$, where E is the brane given by eq. (2.3). This suggests that the brane we are considering is isomorphic to the direct sum of the brane eq. (2.3) and its anti-brane. To verify that two objects are isomorphic, one has to compute the spaces of morphisms between them (in both directions) and check that there exists a pair of even morphisms, going in opposite directions, whose compositions (in either direction) are the identity endomorphisms. We have checked that this is indeed the case. This example illustrates that two very different CDG modules can become isomorphic upon passing to the derived category. In physical terms, very different tachyon profiles can produce the same topological brane.

The case $\beta = y$. The even subspace can be represented by matrices of the form

$$D \in \begin{pmatrix} \mathbb{C}[x]/x^k & \mathbb{C}[x,y]/(x^{n+1} - xy^2) \\ \mathbb{C}[x]/x^{n+1-k} & \mathbb{C}[x]/x^{n+1-k} \end{pmatrix} \subset \mathrm{End}(\mathcal{O}_-^2)$$

that are subject to the divisibility condition

$$x^{n+1} - xy^2 \,\big|\, yx^k(D_{22} - D_{11}) + y^2 D_{12} - x^{2k} D_{21}\,.$$

Solving this algebraic relation gives $2k$ even elements spanning the even subspace of $\mathcal{D}$-cohomology:

$$a_i = \begin{pmatrix} x^i & 0 \\ 0 & x^i \end{pmatrix}, \qquad a_{k+i} = \begin{pmatrix} 0 & x^{i+1} \\ x^{n-2k+1+i} & 0 \end{pmatrix}, \qquad i = 0, 1, \ldots, k-1\,.$$

The odd subspace is computed in the same way and is spanned by the following elements in $\mathrm{Hom}(\mathcal{O}_-^2, \mathcal{O}_+^2)$:

$$\eta_i = \begin{pmatrix} x^{n-2k+1+i} & 0 \\ 0 & x^i \end{pmatrix}, \qquad \eta_{k+i} = \begin{pmatrix} 0 & x^{1+i} \\ x^i & 0 \end{pmatrix}, \qquad i = 0, 1, \ldots, k-1\,.$$

The endomorphism algebra of the brane has two odd generators $\xi = \eta_0$ and $\eta = \eta_k$. The even element a_i is expressed as η^{2i}. The relations are

$$\eta^{2k} = 0\,, \qquad \xi^2 = -\eta^{2n-4k+2}\,, \qquad \xi\eta + \eta\xi = 0\,.$$

Note that the Witten index vanishes for all k. It is also easy to see that all these branes are irreducible, orientable, and pairwise non-isomorphic.

We conjecture that the branes constructed above are the only irreducible branes in D-type minimal models. We expect that this can be proved using the results of ref. [10].

As in the case of A-type minimal models, we can consider orbifolding by the $\mathbb{Z}_2$ symmetry which exchanges branes and anti-branes. We expect that the effect of this is the same as adding a square to the superpotential.

2.4 E-type minimal models

Let us list a few simple examples of branes in E-type minimal models. Our list is not supposed to be exhaustive.

The E_6 minimal model has a Landau-Ginzburg realization with

$$W_{E_6} = x^3 + y^4 \,.$$

Examples of irreducible B-branes are given by

$$E_{k,l} = \left\{ \mathcal{O}_+^2 \oplus \mathcal{O}_-^2 , \;\; Q = \begin{pmatrix} 0 & d_1 \\ d_0 & 0 \end{pmatrix} \right\},$$

where

$$d_0 = \begin{pmatrix} x^k & y^\ell \\ y^{4-\ell} & -x^{3-k} \end{pmatrix}, \qquad d_1 = \begin{pmatrix} x^{3-k} & y^\ell \\ y^{4-\ell} & -x^k \end{pmatrix}.$$

One may choose the fundamental range to be $k = 1$ and $\ell \in \{1, 2\}$. The brane $E_{1,1}$ is orientable, and its anti-brane is $E_{1,3}$. On the other hand, $E_{1,2}$ is unorientable.

The boundary OPE algebras for branes in the fundamental range are as follows. There is a single even generator y which is central. There are also two odd generators ξ and η. The boundary OPE algebra is specified by the following relations

$$\begin{aligned} y^\ell &= \eta^2 = 0 \,, \\ \xi^2 &= -y^{4-2\ell} \,, \\ \xi\eta &= -\eta\xi \,. \end{aligned} \tag{2.6}$$

The E_8 minimal model has a Landau-Ginzburg realization with

$$W_{E_8} = x^3 + y^5 \,.$$

For our purposes, this is very similar to the E_6 case analyzed above. In particular, one immediately concludes that there are two irreducible B-branes labeled by $E_{1,1}, E_{1,2}$. The main difference here is that both of them are orientable now, their anti-branes being given by $E_{1,4}$ and $E_{1,3}$ respectively. The boundary OPE is the same as given in eq. (2.6), except that y^4 must be replaced with y^5 everywhere. Furthermore, in the case of $E_{1,2}$, the even generator y is no longer an independent generator (because $y = -\xi^2$), and the boundary OPE algebra is generated by two odd generators.

The case of E_7 is slightly different. The Landau-Ginzburg superpotential is

$$W_{E_7} = x^3 + xy^3 \,.$$

First of all, there is a brane defined by

$$E = \left\{ \mathcal{O}_+ \oplus \mathcal{O}_- , \;\; Q = \begin{pmatrix} 0 & x^2 + y^3 \\ x & 0 \end{pmatrix} \right\}.$$

This brane is orientable, its anti-brane having the transposed Q. In addition, there are branes specified by higher rank objects:

$$E_\alpha = \left\{ \mathcal{O}_+^2 \oplus \mathcal{O}_-^2 , \;\; Q = \begin{pmatrix} 0 & d_1 \\ d_0 & 0 \end{pmatrix} \right\} ,$$

where

$$d_0 = \begin{pmatrix} x & \alpha \\ \beta & x^2 \end{pmatrix} , \qquad d_1 = \begin{pmatrix} x^2 & \alpha \\ \beta & -x \end{pmatrix} , \quad \alpha\beta = xy^3 .$$

A priori, there are three independent branes associated with the choices $\beta = x, y, y^2$. All of them appear orientable, and their anti-branes are obtained by swapping α and β. However, as in the D-type case, the object with $\beta = x$ is decomposable. In fact one can show that it is isomorphic to the direct sum $E \oplus \bar{E}$, and therefore is unorientable. The objects associated with $\beta = y$ and $\beta = y^2$, on the other hand, are irreducible and orientable. We leave the details to the reader as an exercise.

2.5 Disk correlators in topological minimal models

We will now apply the general formula derived in ref. [8] for topological open string correlators to the branes obtained above. For simplicity, we focus on disk correlators but generalization to higher genera and multi-boundary cases is straightforward. Up to an unessential numerical factor, the disk correlator with a bulk insertion $\alpha \in \mathcal{O}/\partial W$ and a boundary insertion ϕ is given by

$$\langle \alpha \cdot \phi \rangle_{\text{disk}} = \frac{1}{n!\,(2\pi i)^n} \oint_L \frac{\alpha \cdot \mathrm{STr}\left[(\partial Q)^{\wedge n} \phi \right]}{\partial_1 W \, \partial_2 W \ldots \partial_n W} , \tag{2.7}$$

where the integration is carried out over an n-dimensional lagrangian torus. For a more detailed explanation of the formula, see section 1 or ref. [8].

Let us consider an A-type minimal model whose Landau-Ginzburg realization has $W = z^n$. The Jacobi ring is $\mathbb{C}[z]/z^{n-1}$, and it suffices to take the bulk operator to be $\alpha = z^i$ with $i \leq n-2$. As shown in subsection 2.1, irreducible B-branes are labeled by E_k, and it is sufficient to restrict the range of k to $0 < k \leq n/2$. The boundary operator algebra of E_k has an even generator a and an odd generator η, with relations given in eq. (2.2). One easily sees that the supertrace $\mathrm{STr}[\partial Q \cdot \phi]$ vanishes unless $\phi = a^\ell \eta$. In particular, all disk correlators with no boundary insertions vanish. From the closed string point of view, this means that they carry no RR charge. We will come back to this issue when we make comparison with the boundary state formalism in section 3.

When the boundary operator *is* given by $\phi = a^\ell \eta$, our general formula eq. (2.7) yields the following result:

$$\left\langle z^i \cdot a^\ell \eta \right\rangle_{E_k} = \begin{cases} -1 & \text{if } \; \ell + i = k-1 , \\ 0 & \text{otherwise.} \end{cases}$$

As explained in sections 2.1 and 2.2, B-branes in the orbifolded A-type minimal models are the same as B-branes in LG theories with $W = x^n - y^2$. In order to compute disk correlators, we will use the latter description. The usual irreducible B-branes are labeled by $E_{k,\alpha}$ with $0 < k \leq [\frac{n-1}{2}]$ and $\alpha = y$, as listed in (2.3). In case of even n, there are two additional irreducibles $E_{\pm}$. Let us consider the branes $E_{k,y}$ first. The ring of bulk operators is still given by the Jacobi ring $\mathbb{C}[x]/x^{n-1}$. The boundary operator algebras are given in eq. (2.4). An easy computation shows that the supertrace vanishes unless the boundary operator $\xi\eta$ is inserted, so it suffices to consider insertions of the form $x^i \cdot a^\ell \xi\eta$. The disk correlators are computed by the integral

$$\frac{1}{(2\pi i)^2} \oint_L \frac{-2nx^{n+\ell+i-k-1}\, dx \wedge dy}{\partial_1 W\, \partial_2 W} \,,$$

which gives

$$\left\langle x^i \cdot a^\ell \xi\eta \right\rangle_{E_{k,y}} = \begin{cases} 1 & \text{if } \ \ell + i = k-1 \\ 0 & \text{otherwise} \end{cases}$$

Like the branes in the case $W = z^n$ discussed above, these branes carry no RR charge.

Finally let us consider the branes $E_\pm$ which exist when n is even. In the case of E_+, there is a single even generator a with a relation $a^{n/2} = 0$. One can show that the disk correlator for a general insertion $x^i \cdot a^\ell$ is

$$\begin{aligned} \left\langle x^i \cdot a^\ell \right\rangle_{E_+} &= \frac{1}{(2\pi i)^2} \oint_L \frac{-nx^{n/2+\ell+i-1}\, dx \wedge dy}{\partial_1 W\, \partial_2 W} \\ &= \begin{cases} 1/2 & \text{if } \ \ell + i = n/2 - 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The disk correlators for E_- are obtained by a simple sign flip. Notice that in the particular case $\ell = 0$ (no boundary insertion), the nonvanishing disk correlators are

$$\langle x^{n/2-1} \rangle_{E_+} = -\langle x^{n/2-1} \rangle_{E_-} = \frac{1}{2} \,.$$

In this special case disk correlators have also been computed in ref. [4]. Since not all disk correlators without boundary insertions vanish, the branes $E_\pm$ carry nonzero RR charge. There are also RR-charged branes in D-type minimal models, as discussed in the Appendix.

Recall that in the case of even n, the object $E_{n/2,y}$ is reducible and is isomorphic to the direct sum $E_+ \oplus E_-$. The disk correlators computed above are compatible with this assertion. Even though both E_+ and E_- are RR-charged, their direct sum carries no RR charge as $\langle x^i \rangle_{E_+ \oplus E_-} = 0$ for all i. This is in agreement with our earlier statement that $E_{n/2,y}$ is RR-neutral.

This exhausts our list of B-branes in topological A-type minimal models and their $\mathbb{Z}_2$ orbifolds. One can likewise compute disk correlators for B-branes in D-type and E-type minimal models. We give explicit results for the D-type minimal models in the Appendix and leave the E-type to an interested reader.

3. Comparison with the boundary state formalism

In this section we compare our classification of B-branes in $\mathcal{N}=2$ minimal models with the known results obtained from the boundary state formalism. Since, to the best of our knowledge, Cardy branes in D-type $\mathcal{N}=2$ minimal models have not been studied in the literature, we shall limit the discussion to A-type minimal models.

3.1 General remarks

At the outset, we should address one point which may be puzzling at first sight. It seems reasonable to assume that adding squares to the superpotential should not change the low-energy physics, including the topological observables. This is quite obvious in the closed-string sector, because the Jacobi ring is unaffected by the addition of squares. But the situation in the open-string sector is more complicated. It has been shown in ref. [10] that the spectrum of topological branes, as well as the boundary OPE algebra, are unaffected by the addition of *two* squares to W. But we have seen in subsection 2.1 that adding a *single* square to $W=z^n$ has a non-trivial effect on the category of D-branes. In subsection 2.2 we have shown that the effect of adding a single square is the same as the effect of orbifolding the category by a $\mathbb{Z}_2$ symmetry which exchanges branes and anti-branes (at least, in this special case, and probably in general). On the other hand, it is believed that given a superconformal field theory the spectrum of D-branes is uniquely determined. In particular, there should be a unique answer to the question "What is the spectrum of Cardy branes in an $\mathcal{N}=2$ minimal model?"

This apparent conflict between the Landau-Ginzburg and Cardy approaches is resolved by the observation that in fact for every ADE Dynkin diagram there are *two* closely related SCFTs which have the same chiral ring. We will limit our to discussion to A-type Dynkin diagrams. The usual A-type minimal model corresponds to the diagonal combination of characters of the $\mathcal{N}=2$ super-Virasoro algebra. In the notation of ref. [2], its Hilbert space is given by

$$\mathscr{H}=\sum_{(j,n,s)}\mathscr{H}_{j,n,s}\otimes\mathscr{H}_{j,-n,-s}\,.$$

Here the allowed values of the labels j,n,s are

$$\begin{aligned}&j\in\left\{0,\frac{1}{2},1,\dots,\frac{k}{2}\right\},\quad n\in\mathbb{Z}/(2k+4)\mathbb{Z},\quad s\in\mathbb{Z}/4\mathbb{Z}\,,\\&2j+n+s=0\ \mathrm{mod}\ 2\,.\end{aligned}$$

The summation is over all distinct values of (j,n,s), taking into account the following equivalence relation:

$$(j,n,s)\sim\left(\frac{k}{2}-j,n+k+2,s+2\right).$$

The spaces

$$\oplus_{s\,\mathrm{even}}\mathscr{H}_{j,n,s}$$

and

$$\oplus_{s\,\mathrm{odd}}\mathscr{H}_{j,n,s}$$

are irreducible representation of the $\mathcal{N} = 2$ super-Virasoro algebra with central charge $c = 3k/(k+2)$. Even and odd values of s correspond to NS and R sectors, respectively. The variable s is related to the eigenvalue of the fermion number operator F by

$$e^{\frac{i\pi s}{2}} = e^{-\pi i F} .$$

Note that in accordance with this definition F has half-integral eigenvalues in the Ramond sector. The Witten index in the Ramond sector is therefore defined as

$$\mathrm{Tr}_{s\,\mathrm{odd}}(-1)^{F+1/2} .$$

Note also that the above partition function corresponds to the non-chiral GSO projection of type 0A (i.e. opposite projection in the left-moving and right-moving Ramond sectors). We will call this theory the k^{th} minimal model (of type A) and denote it MM_k.

The $\mathbb{Z}_2$ action of interest to us is diagonal in this basis, with eigenvalue $e^{i\pi s}$. Orbifolding by this $\mathbb{Z}_2$ projects out all RR states, but all the twisted sector states are again in the RR sector. The Hilbert space of the orbifolded theory is

$$\mathscr{H}' = \sum_{(j,n,s)} \mathscr{H}_{j,n,s} \otimes \mathscr{H}_{j,-n,s} .$$

This partition function corresponds to the non-chiral GSO projection of type 0B (i.e. the same projection in the left-moving and right-moving Ramond sectors). We will denote this theory $\mathrm{MM}_k/\mathbb{Z}_2$. The two SCFTs are obviously different. But note that the NS-NS sectors in the two theories are the same, hence the chiral rings are also identical. In other words, on a world-sheet without boundaries orbifolding has no effect on topological correlators.[4]

On the other hand, properties of B-branes in the two models are rather different, even in the topological sector [9]. For example, in the unorbifolded $n-2^{\mathrm{nd}}$ minimal model[5] with n even there are $\frac{n}{2}-1$ unorientable irreducible branes with zero Witten index, and two additional branes, which are both orientable, are anti-branes for each other, and have Witten index $n/2$. On the other hand, in the orbifolded $n-2^{\mathrm{nd}}$ minimal model with n even, there are $\frac{n}{2}-1$ orientable branes, their anti-branes, and one more unorientable brane; all these branes have zero Witten index. Comparing with the results of section 2, we see that Cardy branes in the unorbifolded minimal model seem to match topological branes in the LG model $W = x^n - y^2$, while Cardy branes in the orbifolded minimal model match those in the LG model $W = z^n$. Below we will check this identification in more detail. The relation between branes in topological LG models and Cardy B-branes in minimal model has been also discussed in refs. [4, 2] (in some special cases). Our results are in agreement with these papers.

[4]Note that although the two theories have the same chiral ring, the number of RR ground states is different. In fact, there are no RR ground states in the Hilbert space $\mathscr{H}'$. The usual spectral flow argument does not apply here, because $\mathscr{H}'$ is not invariant with respect to spectral flow by $\theta = 1/2$. Despite this, the partition function of the orbifolded theory is modular invariant, as one can easily verify.

[5]Of course, which of the two is the unorbifolded one is a matter of convention. Each of the two models has a $\mathbb{Z}_2$ symmetry, orbifolding by which gives the other model.

3.2 Cardy B-branes in MM_k

The standard Cardy construction leads naturally to A-branes. In particular there is a one-to-one correspondence between RR ground states and Cardy A-branes. To obtain B-branes from a Cardy-like construction, one starts with Cardy A-brane boundary states in the orbifolded model $\mathrm{MM}'_k = \mathrm{MM}_k/\mathbb{Z}_2 \times \mathbb{Z}_{k+2}$, and identifies those which are invariant under $\mathbb{Z}_2 \times \mathbb{Z}_{k+2}$ as B-brane boundary states in the original MM_k. The idea of this construction is that orbifolding by $\mathbb{Z}_2 \times \mathbb{Z}_{k+2}$ acts as a mirror symmetry in this case. The boundary states of these B-branes are constructed in ref. [9]:

$$|j,s\rangle_{\mathrm{B}} = \sqrt{2(k+2)} \sum_{2j',s'\mathrm{even}} \frac{S^{j',0,s'}_{j,-2j-s,s}}{\sqrt{S^{j',0,s'}_{0,0,0}}} |\mathrm{B};j',0,s'\rangle\rangle, \tag{3.1}$$

where $|\mathrm{B};j',0,s'\rangle\rangle$ are the Ishibashi states in MM'_k. Independent Cardy states are parametrized by

$$2j = 0,1,\ldots,\left[\frac{k}{2}\right], \qquad s = 0,1\,.$$

The two choices of s correspond to different B-type supersymmetric boundary conditions $\bar{G}_+ = \pm\bar{G}_-$. We can choose $s=1$ if we fix a sign convention. Note that these boundary states have only NS component. This means that the corresponding branes are unorientable.

Our goal is to compute the spectrum of chiral primary states in the open-string NS sector. It is actually more convenient to compute the number of Ramond ground states, which are obtained from chiral primary states by spectral flow. One can read off the spectrum of Ramond ground states from the boundary state overlaps ${}_B\langle j_1,s_1|q_c^{L_0-c/24}|j_2,s_2\rangle_B$. The even Ramond states can be read off

$${}_{\mathrm{B}}\langle j,0|q_c^{L_0-c/24}|j,1\rangle_{\mathrm{B}} = \sum_{(\tilde{j},\tilde{n})} \left(N^{\tilde{j}}_{jj} + N^{k/2-\tilde{j}}_{jj}\right)\chi_{\tilde{j},\tilde{n},1}(q_o)$$

where

$$N^{\ell}_{ij} = \begin{cases} 1, & \text{if } |i-j| \le \ell \le \min\{i+j,k-i-j\},\ i+j+\ell\in\mathbb{Z} \\ 0, & \text{otherwise}\end{cases}$$

are the fusion coefficients of $\mathrm{SU}(2)_k$.

For our purpose, we are only interested in the multiplicity of ground states, i.e. $(\tilde{j},\tilde{n},\tilde{s}) = (\ell,2\ell+1,1)$. This has a simple form

$$n^{\ell}_{jj} = \begin{cases} 1, & \ell \in \{0,1,\ldots,2j\}\cup\{k,k-1,\ldots,k-2j\} \\ 0, & \text{otherwise}\end{cases}$$

It follows that there are in total $2(2j+1)$ bosonic chiral primary states.

Odd Ramond states can be read off another overlap:

$${}_{\mathrm{B}}\langle j,0|q_c^{L_0-c/24}|j,-1\rangle_{\mathrm{B}}\,.$$

JHEP07(2004)045

In general, $s = 1$ and $s = -1$ boundary states are almost the same, the only difference being the sign of the RR contribution. Therefore in general the number of even and odd open-string states is different. But in the present case, the boundary states have no RR piece, and therefore the $s = 1$ and $s = -1$ boundary states are identical. This implies that the number of odd and even open-string states is the same.

If $k \in 2\mathbb{Z}$, there is an additional subtlety. The above boundary state $|k/4, s\rangle$ is not an irreducible brane, since the multiplicity of the vacuum representation is $n^0_{k/4,k/4} = 2$. Instead it is a sum of two irreducible B-branes, denoted $|k/4, \pm 1\rangle_{\tilde{\mathrm{B}}}$, which are anti-branes of each other. An explicit form of the boundary states can be found in ref. [9], where also the boundary state overlaps have been computed:

$$
\begin{aligned}
{}_{\tilde{\mathrm{B}}}\langle k/4, 0| q_c^{L_0 - c/24} |k/4, 1\rangle_{\tilde{\mathrm{B}}} &= \sum_{j\in\mathbb{Z}} \sum_{n \text{ odd}} \frac{1}{2}\left[1 + (-1)^{j+(n-1)/2}\right] \chi_{j,n,1}\,, \\
{}_{\tilde{\mathrm{B}}}\langle k/4, 0| q_c^{L_0 - c/24} |k/4, -1\rangle_{\tilde{\mathrm{B}}} &= \sum_{j\in\mathbb{Z}} \sum_{n \text{ odd}} \frac{1}{2}\left[1 + (-1)^{j+(n-1)/2}\right] \chi_{j,n,-1}\,.
\end{aligned}
$$

In the first overlap the multiplicity of $\chi_{(j,2j+1,1)}$ is one for all integer j. This shows that there are $k/2+1$ even Ramond ground states. In the second overlap the the multiplicity of $\chi_{(j,-2j-1,-1)}$ is zero for all j. This shows that there are no odd Ramond ground states. The Witten index is $(k+2)/2$ for both the brane and the anti-brane.

Now we want to match topological B-branes in the LG model with $W = x^{k+2} - y^2$ to Cardy B-branes in the minimal model MM_k. We claim that for general k, the Cardy brane $|j, 1\rangle_{\mathrm{B}}$ corresponds to the topological B-brane E_{2j+1} in the LG model specified by the following boundary tachyon profile:

$$
d_0 = \begin{pmatrix} x^{2j+1} & y \\ -y & -x^{k-2j+1} \end{pmatrix}, \qquad d_1 = \begin{pmatrix} x^{k-2j+1} & y \\ -y & -x^{2j+1} \end{pmatrix}
$$

Up to reparametrization, there are $1 + [k/2]$ different E's. This is precisely the number of Cardy B-branes $|j, 1\rangle_{\mathrm{B}}$. The open string spectrum also matches up: the vector space $\mathrm{End}(E_{2j+1})$ has $2(2j+1)$ even basis elements and the same number of odd ones, as we have seen in section 2.

If k is even, then the topological brane $E_{1+k/2}$ is a sum of two irreducible branes. The irreducible branes correspond to the following factorization of W:

$$
Q_\pm = \begin{pmatrix} 0 & x^{1+k/2} \pm y \\ x^{1+k/2} \mp y & 0 \end{pmatrix}.
$$

We identify Q_+ with the Cardy brane $|k/4, 1\rangle_{\tilde{\mathrm{B}}}$ and Q_- with $|k/4, -1\rangle_{\tilde{\mathrm{B}}}$. One can easily check that their open string spectra match up as well.

3.3 Cardy B-branes in $\mathrm{MM}_k/\mathbb{Z}_2$

The B-type Ishibashi states in the orbifold theory are given by A-type Ishibashi states in MM'_k which are fixed by $\mathbb{Z}_{k+2}$. The Cardy states are given by

$$
|j, s\rangle_{\mathrm{B}} = \sqrt{k+2} \sum_{2j'+s'\in 2\mathbb{Z}} \frac{S^{j',0,s'}_{j,-2j-s,s}}{\sqrt{S^{j',0,s'}_{0,0,0}}} |\mathrm{B}; j', 0, s'\rangle\rangle.
$$

The overall normalization factor is fixed by requiring the "vacuum" representation to appear once in the partition function. Independent states come from

$$2j = 0, 1, \ldots, \left[\frac{k}{2}\right], \qquad s = -1, 0, 1, 2\,.$$

By adopting a convention for boundary condition for G, one can restrict the range for s to $s \in \{-1, 1\}$. Note that the boundary states contain RR pieces, and therefore the branes are orientable. The only effect of changing s from 1 to -1 is to change the sign of the RR piece. There is one special case though: for k even and $j = k/4$ the RR piece vanishes, and the values $s = 1$ and $s = -1$ give same brane. Thus the brane with $j = k/4$ is unorientable. This gives a total of $k+1$ branes for any k (counting separately branes and anti-branes).

The even open string spectrum in the Ramond sector is computed by

$$\begin{aligned} Z &= {}_{\mathrm{B}}\langle j, 0|q_c^{L_0 - c/24}|j, 1\rangle_{\mathrm{B}} \\ &= \frac{1}{2}\sum_{2j'+s'\in 2\mathbb{Z}} \frac{S_j^{\,j'} S_j^{\,j'} S_{j'}^{\,\tilde{j}}}{S_0^{\,j'}} \chi_{\tilde{j},\tilde{n},-1}(q_o)\,. \end{aligned}$$

The multiplicity for the Ramond ground state $(j, n, s) = (\ell, -2\ell - 1, -1)$ is given by

$$\begin{aligned} n_{jj}^{\ell} &= \frac{1}{2}\sum_{2j'+s'\in 2\mathbb{Z}} \frac{S_j^{\,j'} S_j^{\,j'} S_{j'}^{\,\ell}}{S_0^{\,j'}} \\ &= \sum_{j'=0}^{k/2} \frac{S_j^{\,j'} S_j^{\,j'} S_{j'}^{\,\ell}}{S_0^{\,j'}} \\ &= N_{jj}^{\ell}\,. \end{aligned}$$

Since that $2j \le k/2$, there is a single Ramond ground state for each ℓ in the range

$$\ell \in \{0, 1, \ldots, 2j\}\,.$$

In other words, there are $2j+1$ even supersymmetric Ramond ground states. We note that the multiplicity of the vacuum representation is always one. This implies that all these B-branes are irreducible.

Similarly the odd Ramond states can be read off from

$${}_{\mathrm{B}}\langle j, 0|q_c^{L_0 - c/24}|j, -1\rangle_{\mathrm{B}} = \frac{1}{2}\sum_{2j'+s'\in 2\mathbb{Z}} \frac{S_j^{\,j'} S_j^{\,j'} S_{j'}^{\,\tilde{j}}}{S_0^{\,j'}} \chi_{\tilde{j},\tilde{n},1}(q_o)\,.$$

The multiplicity of the ground state $(j, n, s) = (\ell, 2\ell + 1, 1)$ is also N_{jj}^{ℓ}. This shows that the number of odd Ramond ground states is also $2j+1$. In particular, the Witten index vanishes for all these branes.

Now we can match the Cardy branes $|j,\pm 1\rangle_{\rm B}$ to topological B-branes in the LG model with $W = x^{k+2}$. This is straightforward:

$$
\begin{aligned}
|j,1\rangle_{\rm B} \leftrightarrow E_{1+2j} &= \left\{ Q = \begin{pmatrix} 0 & x^{2j+1} \\ x^{k-2j+1} & 0 \end{pmatrix} \right\} \\
|j,-1\rangle_{\rm B} \leftrightarrow \bar{E}_{1+2j} &= \left\{ Q = \begin{pmatrix} 0 & x^{k-2j+1} \\ x^{2j+1} & 0 \end{pmatrix} \right\} .
\end{aligned}
$$

Note that for even k and $j = k/4$, $E_{k/2+1}$ is isomorphic to its own anti-brane $\bar{E}_{k/2+1}$, which is compatible with the fact that $|k/4,1\rangle_{\rm B} = |k/4,-1\rangle_{\rm B}$. One can easily see that the open string spectrum computed in section 2 agrees with the CFT computation.

4. Discussion

In this paper we have analyzed the topological sector in $\mathcal{N} = 2$ minimal models with D-branes. We have constructed solutions to the matrix factorization equation, and computed disk correlators. In the case of A-type minimal models, the branes we have constructed exhaust the list of irreducible topological branes, and any topological brane is isomorphic to the direct sum of irreducibles [10]. In the case of D-type minimal models we do not have a proof that we have constructed all irreducible branes, but we suspect that we did. In the case of E-type minimal models we have constructed some examples of irreducible topological branes, but we do not have any intuition about their total number (we expect that it is finite).

We have seen that adding a square to the superpotential W has the same effect on branes as orbifolding by a $\mathbb{Z}_2$ symmetry. In the CFT language, this orbifolding corresponds to changing the GSO projection from 0A to 0B, or vice versa. Although we have demonstrated this only in the case $W = z^n$, we conjecture that the result holds in full generality.

In order to extend the computations in this paper to Gepner models, one has to generalize the formalism to LG orbifolds (and orientifolds). The approach to brane-engineering used in this paper has an obvious “equivariant” version outlined in the end of ref. [8]. What is lacking so far is a formula for topological correlators. In fact, even in the closed string case there is no general formula which would compute topological correlators for LG orbifolds, and existing computations are somewhat ad hoc.

Another interesting direction to explore is the study of gravitational descendants in topological LG models with boundaries. A complete understanding of gravitational descendants would enable one to compute the exact space-time superpotential in Gepner models with D-branes. It would also enable one to understand how the category of D-branes is deformed as one varies the superpotential W.

Acknowledgments

We are grateful to Vladimir Baranovsky, Kentaro Hori, and Dmitri Orlov for useful conversations. This work was supported in part by the DOE grant DE-FG03-92-ER40701.

A. Disk correlators in D-type minimal models

We present here disk correlators in D-type topological $\mathcal{N} = 2$ minimal models for all the B-branes constructed in section 2.3. The algebra of topological closed string states has two generators x and y which satisfy the relations

$$y^2 = (n+1)x^n\,, \qquad xy = 0\,.$$

Clearly one can restrict to monomials $x^i y^j$, with $ij = 0$, $0 \le j \le 1$ and $0 \le i \le n$.

The spectrum of B-branes and the corresponding algebras of boundary operators are analyzed in detail in section 2.3. There we found two general types of branes, which were labeled E and $E_{k,y}$. When $n \in 2\mathbb{Z}$, there are two additional branes which were called E_1 and E_2. See section 2.3 for precise definitions of these objects and their boundary OPE algebras.

Let us start with the brane E. We take the most general operator insertions $x^i y^j \cdot a^\ell$ where $x^i y^j$ is a bulk operator as described above, and a is the even generator of the boundary OPE which we called a_1 in section 2.3. By the results of ref. [8], this disk correlator reduces to a multi-dimensional integral

$$\frac{1}{(2\pi i)^2} \oint_L \frac{-x^i y^{\ell+j+1}\, dx \wedge dy}{\partial_1 W\, \partial_2 W}\,,$$

which can be readily computed by the method of residues. The result is

$$\left\langle x^i y^j \cdot a^\ell \right\rangle_E = \begin{cases} 1 & \text{if } i = 0,\ \ell + j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, there are only two nontrivial disk correlators: $\langle y \rangle_E = \langle a \rangle_E = 1$. All other disk correlators vanish. The disk correlators for the anti-brane $\bar{E}$ are the same, except for a sign flip: $\langle y \rangle_{\bar{E}} = \langle a \rangle_{\bar{E}} = -1$. Both E and $\bar{E}$ carry RR charge.

For the branes labeled by $E_{k,y}$, it can be shown that unless $\xi\eta$ is inserted on the boundary, the supertrace vanishes. One can therefore restrict to insertions of the type $x^i y^j \cdot a^\ell \xi\eta$. Note that a is not an independent generator but is given by $a = \eta^2$. See section 2.3 for various definitions. The disk correlator is given by the integral

$$\frac{1}{(2\pi i)^2} \oint_L \frac{-2(n+1)x^{n+\ell+i-k+1} y^j\, dx \wedge dy}{\partial_1 W\, \partial_2 W}\,,$$

and the result is

$$\left\langle x^i y^j \cdot a^\ell \xi\eta \right\rangle_{E_{k,y}} = \begin{cases} 1 & \text{if } j = 0,\ \ell + i = k - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the only nonvanishing disk correlators are those for $x^i \cdot a^\ell \xi\eta$ that satisfy the selection rule $\ell + i = k - 1$, and they all have equal size. Clearly, these branes do not carry RR charge.

JHEP07(2004)045

Lastly, we have two additional branes E_1 and E_2 if $n \in 2\mathbb{Z}$. The general insertion is $x^i y^j \cdot a^\ell$, and the disk correlators are given by

$$\begin{aligned}\left\langle x^i y^j \cdot a^\ell \right\rangle_{E_1} &= \frac{1}{(2\pi i)^2} \oint_L \frac{x^{\ell+i} y^j \left[(n+1)x^{n/2} - y\right] dx \wedge dy}{\partial_1 W \, \partial_2 W} \\ &= \begin{cases} 1/2 & \text{if } \ i = \ell = 0, j = 1, \\ -1/2 & \text{if } \ j = 0, \ \ell + i = n/2\,, \\ 0 & \qquad \text{otherwise.} \end{cases}\end{aligned}$$

In other words, the only nonvanishing correlators are

$$\langle y \rangle_{E_1} = \frac{1}{2}\,, \qquad \left\langle x^i \cdot a^{n/2-i} \right\rangle_{E_1} = -\frac{1}{2}\,, \qquad i = 0, \ldots, \frac{n}{2}\,.$$

The correlators for other brane E_2 are essentially the same except for a few sign changes:

$$\langle y \rangle_{E_2} = \frac{1}{2}\,, \qquad \left\langle x^i \cdot a^{n/2-i} \right\rangle_{E_2} = \frac{1}{2}\,, \qquad i = 0, \ldots, \frac{n}{2}\,.$$

Recall that we showed earlier that E_1 and E_2 are neither isomorphic objects nor a brane-anti-brane pair. Now we see that this can also be inferred from their disk correlators. If they were isomorphic, their disk correlators would be exactly the same; if they were a brane-anti-brane pair, their correlators would be exactly opposite. Our computation suggests otherwise.

References

[1] I. Brunner, M. Herbst, W. Lerche and B. Scheuner, *Landau-Ginzburg realization of open string TFT*, hep-th/0305133.

[2] I. Brunner and K. Hori, *Orientifolds and mirror symmetry*, hep-th/0303135.

[3] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1994.

[4] K. Hori et al., *Mirror symmetry*, Lectures at the Clay Mathematical School on Mirror Symmetry, Spring 2000, to be published.

[5] K. Hori, A. Iqbal and C. Vafa, *D-branes and mirror symmetry*, hep-th/0005247.

[6] K. Hori, *Linear models of supersymmetric D-branes*, hep-th/0012179.

[7] A. Kapustin and Y. Li, *D-branes in Landau-Ginzburg models and algebraic geometry*, *J. High Energy Phys.* **12** (2003) 005 [hep-th/0210296].

[8] A. Kapustin and Y. Li, *Topological correlators in Landau-Ginzburg models with boundaries*, *Adv. Theor. Math. Phys.* **7** (2004) 727 [hep-th/0305136].

[9] J.M. Maldacena, G.W. Moore and N. Seiberg, *Geometrical interpretation of D-branes in gauged WZW models*, *J. High Energy Phys.* **07** (2001) 046 [hep-th/0105038].

[10] D. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, math.AG/03023040302304.

[11] C. Vafa, *Topological Landau-Ginzburg models*, *Mod. Phys. Lett.* **A 6** (1991) 337.

JHEP07(2004)045